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AN INTRODUCTION TO THE REGULAR THEORY OF FAIRNESS *

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Abstract. We present an approach to fairness in the style of the theory of ω-regularity. Several concepts of fairness are researched and compared within the classical language theory of words of infinite length.

1. Introduction

In this paper we present an attempt to attack the phenomenon of fairness in a framework of formal language and automata theory. Therefore, we will not introduce a new adjusted fairness concept but will deal with the known fairness phenomenon from problems of distributed asynchronous or nondeterministic systems. The concept of fairness has attracted much attention in recent years in studies of the behavior of nondeterministic or concurrent systems or programs. Fairness and several closely related concepts have been introduced, such as starvation-freeness, justice, weak-, strong- and extreme-fairness, S-T-P-fairness, fair termination, etc. The common idea is that any component of some system that is enabled sufficiently often must eventually be activated.

Unfortunately, as most research on fairness and its relatives is done in quite different specific models, it becomes difficult to distinguish between results about fairness itself and about the underlying model. Some work has been done inside Petri-net theory. For instance, Best [3] presented a hierarchy of n-fairness that collapses in simple nets. Fairness has also been studied in CCS (see [16]) and CCS-like calculi, see, e.g., [6], where a calculus is presented for exactly the fair expressions in a CCS-like language, or [7] with an abstract model of fair computations. Most work on fairness has been done in programming languages, see, e.g., [1, 8, 10, 12, 15], where fairness is also connected with temporal logic. A quite general approach was done by Queille and Sifakis [22], where fairness is studied in T-systems. In the style of our following approach a first result, namely that ω-regular languages are closed under fair merge, can be found in Park [18]. A book

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on fairness by Francez [11] has recently been published. It presents a very good introduction into the problems connected with fairness and treats fairness with respect to programming.

As fairness is inherently noncontinuous and has many connections to probability theory and temporal logic, a treatment within the framework of classical automata and formal language theory seems not to be too promising. However, we will show that fairness can be embedded into these classical theories very smoothly with many quite general results. An introduction into our approach with many detailed proofs is presented here. We use multigraphs or automata as a basic model where we can distinguish between actions (transitions, arcs) and their names. Also we restrict our research in this paper on finite structures, i.e., the regular case is studied. Our main purpose is to get deeper insight in these fairness phenomena without sticking too close to a specific model (such as some concurrent language). However, such general results should later lead to conclusions for applied computer science.

A first version of this paper without any proof was presented at *STACS 87* [21].

2. Basic definitions

We frequently use standard mathematical abbreviations. For instance, $\forall x(p(x))$: ... reads: for all x with the property $p(x)$ it holds ...; $\exists x$: ... reads: there exists an x s.t. ....

For any set A the cardinality of A is denoted $|A|$. The powerset of A is written as 2^A. $\mathbb{N}$ is the set of all positive, $\mathbb{N}_0$ the set of nonnegative integers. For set inclusion we use $\subseteq$, and $\subset$ for proper set inclusion.

For an alphabet Σ let Σ^* (Σ^ω respectively) denote the set of all finite (infinite) sequences or *words* over Σ (Σ^* is the free monoid). The *empty sequence* is denoted λ, $\Sigma^+ := \Sigma^* - \{\lambda\}$. For an alphabet Σ and $\varepsilon \notin \Sigma$ let $\Sigma_\varepsilon := \Sigma \cup \{\varepsilon\}$. $|w|$ denotes the length of a word w, so if $w = x_1 \ldots x_n \in \Sigma^*$ ($x_i \in \Sigma$ for $i \in \{1, \ldots, n\}$), $|w| := n$ and if $w \in \Sigma^\omega$, we let $|w| := \omega$ where $\omega > n$ for all $n \in \mathbb{N}$. The empty sequence has length 0.

We identify words $w \in \Sigma^*$ with sequences of letters of Σ. $w(i)$ is the ith element of the sequence w if $|w| \geq i$, and remains undefined otherwise. With $w[i]$ we denote the prefix of length i, so we let $w[i] := w(1) \ldots w(i)$ if $|w| \geq i$ and $w[i] := w$ otherwise.

The set of arbitrary long sequences is $\Sigma^\infty := \Sigma^* \cup \Sigma^\omega$. For $v \in \Sigma^*$ and $w \in \Sigma^\infty$ the concatenation $v \circ w$ (usually written as vw if no confusion will arise) is defined as the word u that is uniquely determined by $u(i) = v(i)$ for $i \leq |v|$ and otherwise $u(i) = w(i - |v|)$. For $v \in \Sigma^\omega$ and $w \in \Sigma^\infty$, vw remains undefined.

An *ω-language* (*∞-language*, **-language* respectively) is a subset of Σ^ω (Σ^∞, Σ^*). In the sequel we will use the word *language* to denote a *-language, ω-language or ∞-language.

For $v \in \Sigma^*$, v^ω denotes the string $u \in \Sigma^\omega$, where, for all $i \leq |v|$ and for all $n \in \mathbb{N}_0$, $u(n|v| + i) = v(i)$. Let $(w_i)_{i \in \mathbb{N}}$ be a sequence of finite words on Σ. We write $u = w_1 \ldots w_n \ldots$ for the uniquely determined word $u \in \Sigma^\omega$ determined by the equation $u[\sum_{i=1}^{n} |w_i|] = u[\sum_{i=1}^{n-1} |w_i|] \circ w_n$, that is, the successive concatenation of the words w_i.

On Σ^∞ we define a partial order $\leqslant$ by setting $v \leqslant w$ iff there exists an $i \in \mathbb{N}$ s.t. $v = w[i]$. We will write $v < w$ if $v \leqslant w$ and $v \neq w$. If $v \leqslant w$ $(v < w)$, v is a (proper) *prefix* of w. The set of all prefixes of a word w is abbreviated as $\mathrm{Pref}(w)$, i.e., $\mathrm{Pref}(w) := \{u \in \Sigma^* \mid u \leqslant w\}$. For a sequence $(w_i)_{i\in\mathbb{N}}$ of words in Σ^* with the property $w_i < w_{i+1}$ we define $w := \lim_{i\to\infty} w_i$ to be the word $w \in \Sigma^\omega$ s.t. $\forall i \in \mathbb{N}$: $w_i < w$. For $v, w \in \Sigma^\infty$ we say v is a *subword* of w ($v \subseteq w$ for short) iff $\exists u \in \Sigma^*$ s.t. $uv \leqslant w$. If a subword occurs infinitely often, i.e., if $\exists J \subseteq \mathbb{N}\ |J| = \omega$ s.t. $\forall j \in J$: $w[j]v < w$, then we will write $v \subseteq_\omega w$. If $|v| = 1$, we sometimes write $v \in w (v \in_\omega w)$ instead of $v \subseteq w$ $(v \subseteq_\omega w)$.

For any language M over Σ we define the following sets:

(1) $\mathrm{Pref}(M) := \bigcup_{w\in M} \mathrm{Pref}(w)$, the *prefix language* of M. By the way, note that $\mathrm{Pref}(M)$ is always a $*$-language, i.e., a language in the classical sense.

(2) The set $\bar{M}$ is called the *closure* or *ω-closure* of M and is defined as

$$\bar{M} := \{w \in \Sigma^\omega \mid \exists (w_i)_{i\in\mathbb{N}}\colon \forall i\colon w_i \in M \text{ and } w_i < w_{i+1} \text{ and } w = \lim_{i\to\infty} w_i\}.$$

(3) Further the *adherence* of M is $\mathrm{adh}(M) := \overline{\mathrm{Pref}(M)}$.

(4) Additionally, for any $*$-language M the *infinite iteration* is defined as

$$M^\omega := \{w \in \Sigma^\omega \mid \exists (w_i)_{i\in\mathbb{N}}\colon w_i \in (M - \{\lambda\})^i \text{ and } w_i < w_{i+1} \text{ and } w = \lim_{i\to\infty} w_i\}$$

for $M \not\subseteq \{\emptyset, \{\lambda\}\}$ and $\emptyset^\omega := \{\lambda\}^\omega := \{\lambda\}$. Also we let $M^\infty := M^\omega \cup M^*$.

3. Automata

Within this section we will introduce our model of computation, the multigraph, which is similar to notions like 'automaton' or 'transition-system'.

Definition 3.1. A *finite multigraph* G is a 3-tuple $G = (V, E, \mu)$ of finite sets V (of vertices), E (of edges), and a mapping $\mu : E \to V \times V$. If $\mu(e) = (v_1, v_2)$, we say that e is an edge from vertex v_1 to v_2.

A *path* p in G is a finite or infinite sequence $(e_i)_{i\in I}$ of edges $e_i \in E$ s.t. $\mu(e_i) = (v_i, v_{i+1})$, $\mu(e_{i+1}) = (v'_{i+1}, v'_{i+2})$ implies that $v_{i+1} = v'_{i+1}$ $\forall i, i+1 \in I$, where $I = \mathbb{N}$ or $I = \{i \mid 1 \leqslant i \leqslant k\}$ for some $k \in \mathbb{N}$. We thus regard a path as a word $p \in E^\infty$.

We use the standard technical terms of graphs. As $\mu(e_1) = \mu(e_2)$ for $e_1 \neq e_2$ may hold, we allow multiple arcs between vertices.

Definition 3.2. An *automaton* A is a 6-tuple $A = (S, \Sigma, E, \mu, \Phi, s_A)$ of finite sets S (of states), Σ (of actions), and E of edges s.t. (S, E, μ) is a multigraph with a weight-function $\Phi : E \to \Sigma_\varepsilon$ and an initial state $s_A \in S$.

An ε-edge (i.e., an edge e with $\Phi(e) = \varepsilon$) will be mapped onto the empty word λ by Φ^λ. Therefore, we extend Φ to the fine homomorphism $\Phi : E^\infty \to \Sigma_\varepsilon^\infty$ as usual via:

$$\Phi(\lambda) := \lambda \quad \text{and} \quad \Phi(ep) := \Phi(e)\Phi(p) \quad \text{for } e \in E \text{ and } p \in E^{\infty}.$$

From this we derive the homomorphism $\Phi^{\lambda}: E^{\infty} \to \Sigma^{\infty}$ by setting $\Phi^{\lambda}(e) := \Phi(e)$ if $\Phi(e) \in \Sigma$ and $\Phi^{\lambda}(e) := \lambda$ if $\Phi(e) = \varepsilon$.

In the sequel the initial state of an automaton A will always be denoted s_A.

In contrast to the classical theory of automata we make a clear distinction between edges $e \in E$ and their names $\Phi(e) \in \Sigma_{\varepsilon}$. Furthermore, it is sometimes important to be able to talk about ε-edges in a different manner than about λ. That is possible in this model because of the use of two morphisms Φ and Φ^{λ}.

Infinite automata (with an infinite set of states) are a very general model, able to describe T-Systems, Petri-nets, communicating systems of Hoare [14] and Milner's CCS [16] (in part, if one does not want to use the recursion operator to get dynamically growing automata). Parallel activities are easily described in this model in the standard manner using a diamond construction as the 'process' of a parallel system is such an infinite automaton.

Our main restriction is that we will deal only with *finite* automata in this paper. Thus we try to develop the regular part of a theory of fairness here.

Definition 3.3. Let $A = (S, \Sigma, E, \mu, \Phi, s_A)$ be an automaton. A path p in A is a path in the included multigraph. Let $p = (e_i)_{i \in \mathbb{N}} \in E^{\omega}$ be an infinite path, $p' = (e'_i)_{0 \leq i \leq n}$ be a finite path, with $\mu(e'_i) = (s_i, s_{i+1})$ We say

- p' *starts from* s_0, *passes* s_i $(i \in \{0, \dots, n+1\})$ and ends in s_{n+1};
- p' is *final* iff $\forall e \in E$: $p'e$ is no path in A;
- p' is a *subpath* of p iff p' is a subword of p; we use the notations for words also for paths;
- p *passes* a state $s \in S$ *infinitely often* iff

$$\exists J \subseteq \mathbb{N}: |J| = \omega \text{ s.t. } \forall j \in J: \exists s_j \in S: \quad \mu(e_j) = (s_j, s).$$

The *infinity set* of p is the set

$$S^{\omega}(p) := \{s \in S \mid p \text{ passes } s \text{ infinitely often}\}.$$

Analogously, we define

$$E^{\omega}(p) := \{e \in E \mid e \in_{\omega} p\}.$$

We will need these preliminary definitions of ∞-languages and multigraphs to define fairness, as the property 'fairness' will state that 'some action that may be done infinitely often will be done infinitely often'. This general idea will be made precise on the abstract level of multigraphs now.

4. Fairness

Definition 4.1. Let $A = (S, \Sigma, E, \mu, \Phi, s_A)$ be an automaton, $e \in E$ and $p \in E^\omega$ be an infinite path. The path p is called:
- *edge-fair* (ef) iff

$$\forall s \in S^\omega(p): \forall e \in E: \quad (\exists s' \in S: \mu(e) = (s, s') \Rightarrow e \in_\omega p);$$

- *path-fair* (pf) iff

$$\forall s \in S^\omega(p): \forall \text{paths } p_0 \in E^+: \quad (p_0 \text{ starts from } s \Rightarrow p_0 \subseteq p);$$

- *letter-fair* (lf) iff

$$\forall s \in S^\omega(p): \forall e \in E: \quad (\exists s' \in S: e = (s, s') \Rightarrow \Phi(e) \in_\omega \Phi(p));$$

- *word-fair* (wf) iff

$$\forall s \in S^\omega(p): \forall \text{paths } p_0 \in E^+: \quad (p_0 \text{ starts from } s \Rightarrow \Phi(p_0) \subseteq \Phi(p)).$$

Further, every final path p is called x-fair for every $x \in \{\text{edge, path, letter, word}\}$.

In constrast to edge- and letter-fairness it suffices to state $p_0 \subseteq p$ ($\Phi(p_0) \subseteq \Phi(p)$) for path- and word-fairness as this implies the $\subseteq_\omega$ relation obviously. Notice that in the definition of lf and wf paths we do not use the mapping Φ^λ, as we wish ε-edges to be treated fair as well.

Definition 4.2. Let $A = (S, \Sigma, E, \mu, \Phi, s_A)$ be an automaton. The *languages* induced by A are defined as follows:

$$L(A) := \{w \in \Sigma^* \mid \exists p \in E^* \text{ a final path starting in } s_A \text{ s.t. } w = \Phi^\lambda(p)\},$$

$$L^{xf}(A) := \{w \in \Sigma^\infty \mid \exists \text{path } p \in E^\infty \ x\text{-fair, starting in } s_A \text{ s.t. } w = \Phi^\lambda(p)\},$$

where xf denotes one of the fairness notions from above.

For $s, t \in S$, let

$$L_{s,t}(A) := \{w \in \Sigma^* \mid \exists p \text{ path from } s \text{ to } t\text{: } w = \Phi^\lambda(p)\}.$$

For $F \subseteq S$, let

$$L^\omega(A, F) := \{w \in \Sigma^\omega \mid \exists \text{path } p \in E^\omega \text{ starting in } s_A, \\ F \cap S^\omega(p) \neq \emptyset \text{ s.t. } w = \Phi^\lambda(p)\},$$

$$L^\infty(A, F) := L^\omega(A, F) \cup L(A).$$

For $^o \in$ {edge-fair, letter-fair, path-fair, word-fair} we define Rec^o_Σ to be the set of all languages M s.t. there exists an automaton A with $M = L^o(A)$. $\mathrm{Rec}^\omega_\Sigma$ is the set of all languages M s.t. there exists an A and F s.t. $M = L^\omega(A, F)$.

Usually, the recognizable languages are those accepted by automata with final states. We do not need these because any automaton with ε-edges only needs to have one single final state, and this state may be chosen to be a sink state. This is not true for ω-regular languages, where final states are required. However, we will be able to prove our main theorem (Rec and Rat coincide for some fairness conceptions; see Theorem 7.6) without any need for final states.

To get an idea of the fair languages of an automaton let us have a look at some very simple examples.

Examples. For the automata A_i of Fig. 1 it holds that:

(1) $L(A_1)=\emptyset$, $L^{\mathrm{ef}}(A_1)=L^{\mathrm{lf}}(A_1)=(a^*bb^*a)^\omega$ but $(a^*bb^*a)^\omega \neq L^{\mathrm{pf}}(A_1)\cup L^{\mathrm{wf}}(A_1)$ as, for instance, the word $(ab)^\omega \notin L^{\mathrm{pf}}(A_1)\cup L^{\mathrm{wf}}(A_1)$. In $(ab)^\omega$ not every enabled subpath (or subword) is used!

(2) $L^{\circ}(A_2)=(a+b)^*$ for $^{\circ}\in\{\emptyset, \mathrm{ef}, \mathrm{lf}, \mathrm{pf}, \mathrm{wf}\}$

(3) $L^{\circ}(A_3)=(a+b)^*a$ for $^{\circ}\in\{\emptyset, \mathrm{ef}, \mathrm{pf}\}$, but $(a+b)^*a$ is properly included in the letter-fair as well as in the word-fair language as these also include infinite words since such a path may always use the a-edge from s_1 to s_1 instead of that to s_2!

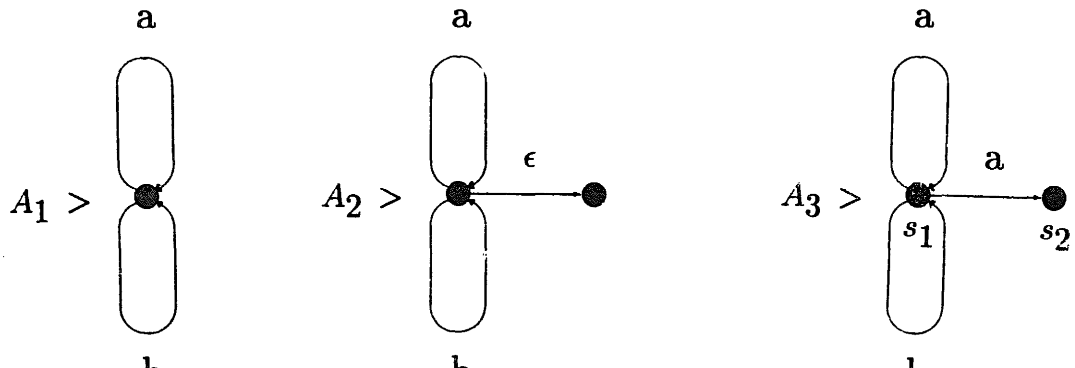
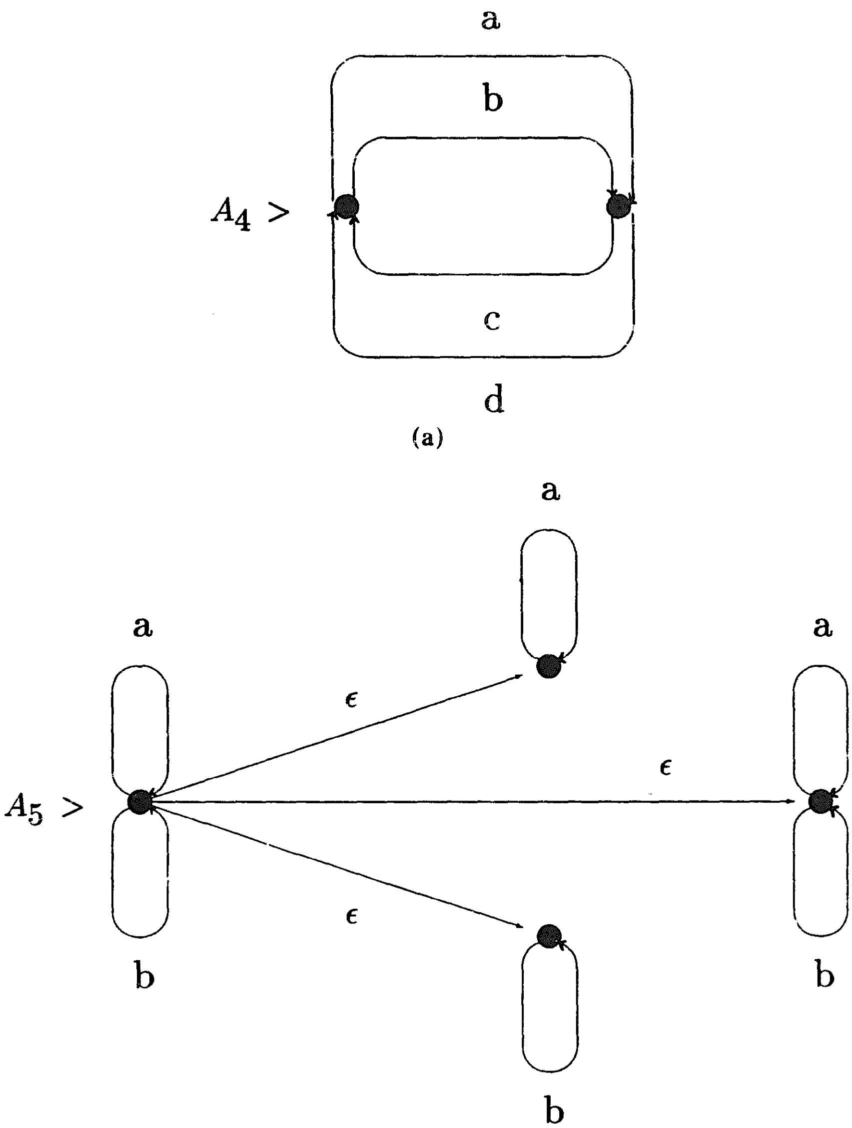


Fig. 1.

A_4 (see Fig. 2) clearly shows the differences between edge- or letter-fairness on the one hand and path- or word-fairness on the other hand. The 'deterministic' process $(acbd)^\omega$ is a word in $L^{\mathrm{ef}}(A_4)$ and $L^{\mathrm{lf}}(A_4)$ but not in $L^{\mathrm{pf}}(A_4)$ or $L^{\mathrm{wf}}(A_4)$! Thus path- and word-fairness force a certain randomness in the behavior.

pf and wf are closely related to P-fairness of Queille and Sifakis [22] and ∞-fairness of Best [3], whereas ef and lf are related to strong fairness (of several authors) and to S- and T-fairness of Queille and Sifakis. In fact, T-fairness in finite models is our edge-fairness.

A_5 is quite interesting: Note that $L^{\mathrm{ef}}(A_5)=L^{\mathrm{lf}}(A_5)=\{a,b\}^\omega$. On the other hand, $\Sigma^\omega\notin \mathrm{Rec}_\Sigma^{\mathrm{pf}}\cup\mathrm{Rec}_\Sigma^{\mathrm{wf}}$ for $|\Sigma|>1$, as is easily seen. (The reader may try to give a proof as an exercise.) A_5 is a first hint that treatment of edge- and letter-fairness will differ from a theory of path- and word-fairness.



(a)

(b)

Fig. 2.

Lemma 4.3. *Let* xf $\in$ {ef, lf, pf, wf} *and A be an automaton,*

(i) $L^{\mathrm{pf}}(A) \subseteq L^{\mathrm{wf}}(A) \cap L^{\mathrm{ef}}(A)$;

(ii) $L^{\mathrm{ef}}(A) \subseteq L^{\mathrm{lf}}(A)$;

(iii) $\mathrm{Rec}_\Sigma \subseteq \mathrm{Rec}_\Sigma^{\mathrm{xf}}$.

Proof. (i) and (ii) are obvious.

(iii) Let $M \in \mathrm{Rec}_\Sigma$ be any recognizable *-language. There exists a trim ε-free automaton $A = (S, \Sigma, E, \mu, \Phi, s_A)$, a set $F \subseteq S$ for A s.t.

$$M = \{w \in \Sigma^* \mid \exists \text{path } p \text{ from } s_A \text{ to some final state of } F \text{ s.t. } w = \Phi^\lambda(p)\}.$$

(trim means that any state $s \in S$ is passed by some path from the initial state to some final state; ε-free means that $\Phi(e) \neq \varepsilon\ \forall e \in E$). We transform A into some automaton A' with a new sink state s_f and some additional states by the following

method: For every state $s \in S$ choose some paths q_S leading from s to some final state $s_f' \in F$. For $|q_S| = n$ choose n new auxiliary states $(s, 1), \ldots, (s, n)$ and connect them as follows by:
- an edge e_0 with $\mu(e_0) = (s, (s, 1))$ and $\Phi(e_0) = \varepsilon$;
- edges e_i with $\mu(e_i) = ((s, i), (s, i+1))$ and $\Phi(e_i) = \Phi(q_S)(i)$ for $1 \leq i \leq n$;
- an edge e_n with $\mu(e_n) = ((s, n), s_f)$ and $\Phi(e_n) = \Phi(q_S)(n)$;
- for $n = 0$ connect s directly with s_f with an ε-edge.

Define

$$M' := \{w \in \Sigma^* \mid \exists \text{path } p \text{ in } A' \text{ from } s_A \text{ to } s_f \text{ s.t. } w = \Phi^\lambda(p)\}.$$

Obviously, $M' = M$.

It suffices to prove that $M' = L^{xf}(A')$ for $xf \in \{\text{lf}, \text{ef}, \text{wf}, \text{pf}\}$. Therefore, it suffices to prove that any xf path in A' must eventually lead to s_f, However, this is obvious as from any state of A an ε-edge leaves. Thus, any xf-path has to use such an ε-edge eventually. Once an ε-edge is used, it only leads to s_f by the above construction. □

It should be noted that, for word-, edge- or path-fairness, a much more simple automaton A' may be constructed. However, the above proof holds for all fairness concepts. This method of introducing finite paths without connections to the original graph beside its two endpoints will be used quite frequently.

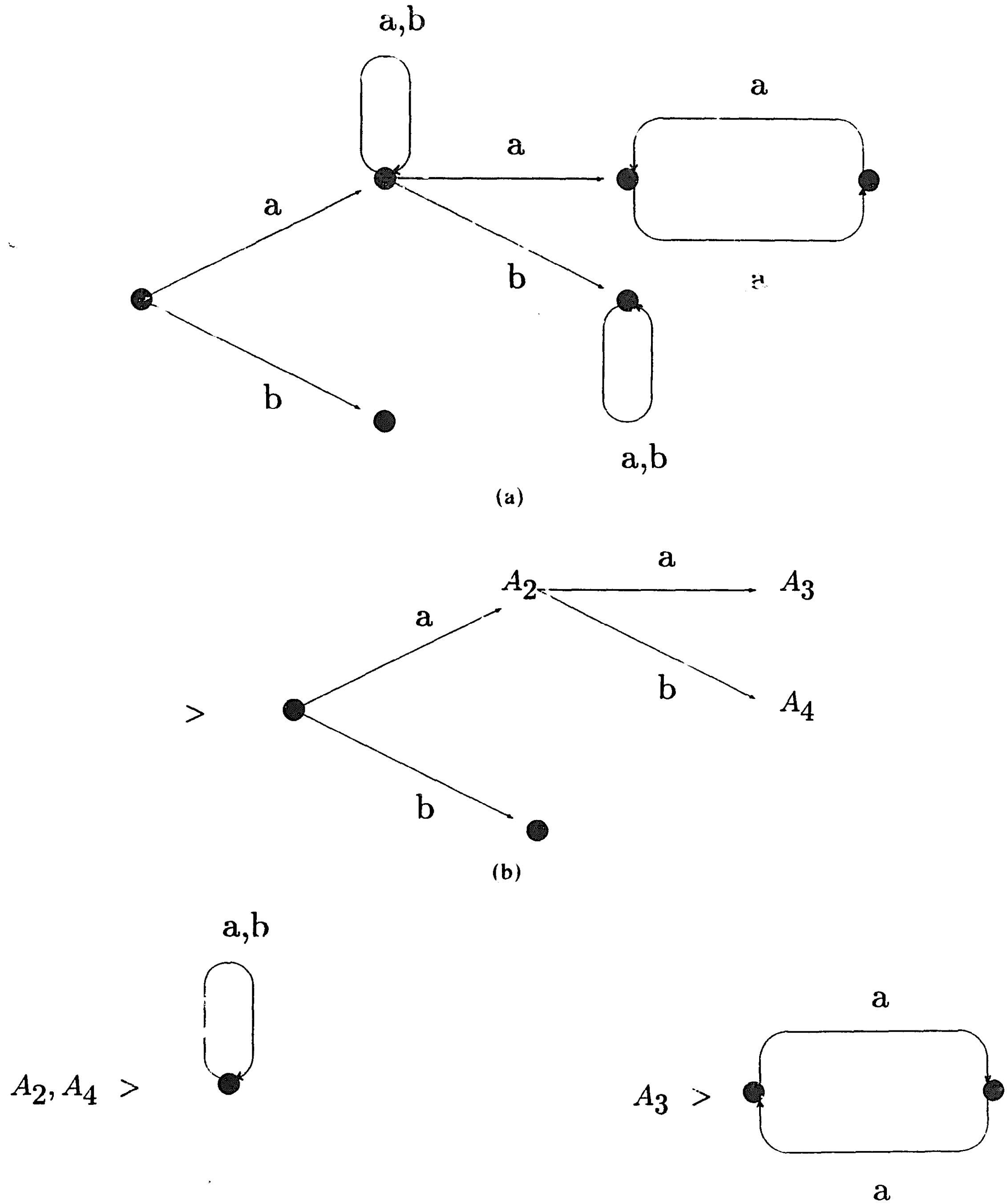
The proof of the previous lemma shows why we introduce ε-edges and a different symbol for the empty word. ε is regarded as a usual letter in the definition of fair paths, but becomes mapped onto λ through Φ^λ. We need this ability to name an edge e with $\Phi^\lambda(e) = \lambda$ if we want moves of the automata to be ignored in terms of the induced language. In contrast to path- and edge-fairness, where ε-edges are not necessary, these edges are important if one deals with word- or letter-fairness; otherwise, e.g., $\{a\}^*$ could not be a word-fair language! With a standard trick one may eliminate the ε-edges in the above construction without changing the path- or edge-fair languages.

5. Normal forms

The concept of fair words in the regular case has one striking property, which leads to the following normal forms for fair languages. There is a strong relation between finite paths and final states on the one hand and infinite fair paths and certain subautomata (called final) on the other. Consider the following example.

Example (*cf. Fig.* 3). For the automata of Fig. 3 it holds that

$$\begin{aligned} L^{\text{ef}}(A) &= b + a\{a, b\}^* a L^{\text{ef}}(A_3) + a\{a, b\}^* b L^{\text{ef}}(A_4) \\ &= b + a\{a, b\}^* a a^\omega + a\{a, b\}^* b L^{\text{ef}}(A_4); \\ L^{\text{pf}}(A) &= b + a\{a, b\}^* a L^{\text{pf}}(A_3) + a\{a, b\}^* b L^{\text{pf}}(A_4) \\ &= b + a\{a, b\}^* a a^\omega + a\{a, b\}^* b L^{\text{pf}}(A_4). \end{aligned}$$



(c)
Fig. 3.

But,

$$L^{\mathrm{lf}}(A) = b + a\{a, b\}^* L^{\mathrm{lf}}(A_2) + a\{a, b\}^* aL^{\mathrm{lf}}(A_3) + a\{a, b\}^* bL^{\mathrm{lf}}(A_4);$$

$$L^{\mathrm{wf}}(A) = b + a\{a, b\}^* L^{\mathrm{wf}}(A_2) + a\{a, b\}^* aL^{\mathrm{wf}}(A_3) + a\{a, b\}^* bL^{\mathrm{wf}}(A_4).$$

as a lf- or wf-path may stay in A_2 forever.

Note that there is a trivial representation of $L^{\mathrm{xf}}(A_3)$ as a^ω, whereas there is nothing known about $L^{\mathrm{xf}}(A_4)$ yet. (This is treated in Sections 6 and 7). Nevertheless, this example shows that all infinite xf paths end in particular strongly connected subautomata. Therefore, it is important to inquire these special strongly connected components, which may vary for the various notations of fairness as ef, lf and wf.

Because an infinite xf path does not leave a certain subautomaton (if we consider the underlying hyperstructure), we will call these subautomata edge-, path-, letter- and word-final. Furthermore, we regard a sink state as a trivial final subautomaton consisting only of this state and no arcs. This leads to the following definition.

Definition 5.1. Let $A=(S, \Sigma, E, \mu, \Phi, s_A)$ be an automaton and $xf \in \{\text{ef}, \text{lf}, \text{wf}, \text{pf}\}$. For $x \in \{\text{edge, path, letter, word}\}$, an *x-final subautomaton* of A is a subautomaton $A'=(S', \Sigma, E', \mu', \Phi', s_{A'})$ s.t. $\mu'=\mu_{/A'}$, $\Phi'=\Phi_{/A'}$ and

(i) (*finite case*)

$$S'=\{s_f\} \quad (s_f \text{ sink state in } A), \qquad E'=\emptyset, \qquad s_{A'}=s_f$$

(such subautomata will be called *trivial*), OR

(ii) (*infinite case*) $\exists p$: $p\,xf$ path in A, $|p|=\infty$ s.t.

$$S'=S^{\omega}(p), \qquad s_{A'} \in S', \qquad E'=E^{\omega}(p) \quad \text{and} \quad \Phi^{\wedge}(p) \in \Sigma^{\omega}.$$

Define: $\text{Fin}_{xf}(A) := \{A' \mid A' \text{ } x\text{-final subautomaton of } A\}$.

Note that in the definition above $\Phi^{\wedge}(p)$ is an infinite word. For an algebraical treatment of path- and word-fairness it is necessary to avoid that a path- or word-final subautomaton consists only of ε-edges, as shown in Fig. 4.

In Fig. 4 an infinite path- or word-fair path generates a finite fair word. Without loss of generality, we substitute these subautomata by an ε-edge leading to a sink state, which leads to the automata of Fig. 5.

Thus, in every nontrivial path- or word-final subautomaton there exists an edge e s.t. $\Phi(e) \neq \varepsilon$.

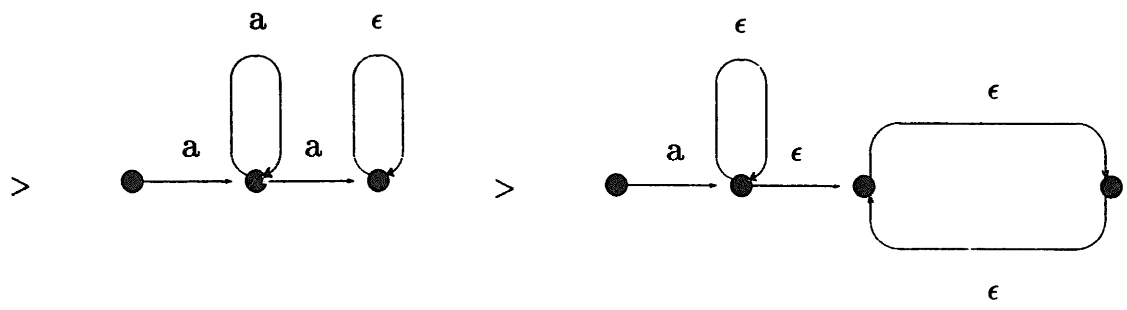


Fig. 4.

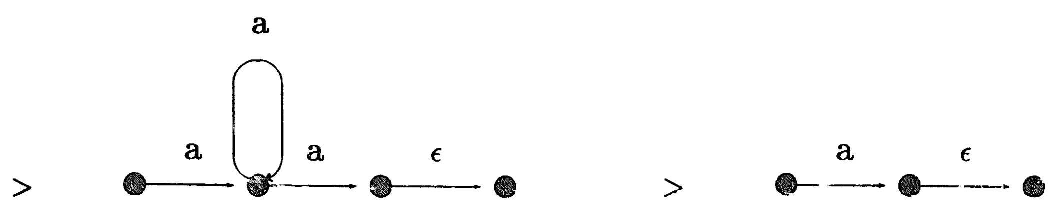


Fig. 5.

It is not hard to see that the set of all edge-final subautomata of a given automaton A is identical to the set of all path-final subautomata. This is not true for all other combinations of our fairness definitions. Therefore, we denote edge- and path-final subautomata generally as *final*. The following corollary gives another representation for final and word-final subautomata of an automaton A where the specific ε-loops are substituted in a manner as discussed above.

Corollary 5.2. *A nontrivial subautomaton $A_f = (S_f, \Sigma, E_f, \mu_{/A_f}, \Phi_{/A_f}, s_f))$ of a trim automaton $A = (S, \Sigma, E, \mu, \Phi, s_A)$ is*
(1) final *iff*
 (i) *A_f is trim, i.e., $\forall s \in S_f$: $\exists p, q$ paths in A_f:*

$$p \text{ leads from } s_f \text{ and } s, \qquad q \text{ leads from } s \text{ to } s_f$$

 (ii) $\forall p \in E^*$: $\forall s_1, s_2 \in S$:

$$\text{path } p \text{ leads from } s_1 \text{ to } s_2 \text{ and } s_1 \in S_f \Rightarrow s_2 \in S_f;$$

(2) word-final *iff*
 (i) *holds*
 (ii′) $\forall p \in E^*$: $\forall s_1, s_2 \in S$:

$$\text{path } p \text{ leads from } s_1 \text{ to } s_2 \text{ and } s_1 \in S_f \Rightarrow \exists \text{path } q \in E_f^*: \Phi(q) = \Phi(p).$$

Proof. ($\Rightarrow$): (1): Let A be a path-final subautomaton of A. If A_f consists of a single final state, conditions (i) and (ii) are valid. If there exists a path-fair path p: $|p| = \omega$ s.t. $S_f = S^\omega(p)$ and $E_f = E^\omega(p)$, then from $S_f = S^\omega(p)$, we get condition (i). The fairness assumption for p forces that every reachable (from a state of S_f) state $s \in S$ must be reached infinitely often s.t. condition (ii) holds.

(2): Let A be a word-final subautomaton of A s.t. there exists a word-fair path p: $|p| = \infty$ s.t. $S_f = S^\omega(p)$ and $E_f = E^\omega(p)$. (i) is proved analogously to (1). In contrast to (1), the word-fairness assumption for p does not force a path to be taken infinitely often. It forces only that $\Phi(q)$, for a path q that could be chosen infinitely often, occurs infinitely often in $\Phi(p)$, which implies condition (ii′).

($\Leftarrow$): (1): Let A_f be as described by conditions (i) and (ii). Construct a path-fair path p in A with $A_f = (S^\omega(p), \Sigma, S^\omega(p), \mu', \Phi', s_f)$ as follows: As A is trim we get $\exists p'$: p' path from s to s_f. Let $p_{n,j}, j = 1, \ldots, j_n$, be the paths of length n in A_f starting from s_f. From condition (i) it follows that $\forall p_{n,j}\ \exists q_{n,j}$: $p_{n,j}q_{n,j}$ is a path from s_f to s_f. Then, define

$$p := p_{1,1}q_{1,1}p_{1,2}q_{1,2} \cdots p_{2,1}q_{2,1}p_{2,2}q_{2,2} \cdots p_{2,j_2}q_{2,j_2} \cdots,$$

an infinite path in A_f with $S^\omega(p) = S_f$ and $E^\omega(p) = E_f$. Assume p is not path-fair in A_f. Then, $\exists s \in S_f = S^\omega(p)$, q path from s to some state $t \in S$, s.t. q does not occur infinitely often in p. As $s \in S_f$ and with condition (ii), it holds that $t \in S_f$. Thus, $\exists p''$ s.t. $p''q$ is a path from s_f, whence $p''q$ occurs infinitely often in p. So q occurs infinitely often in p. Thus, p is a path-fair path in A which does not leave A_f.

(2): Analogously. □

Lemma 5.3. *Let $A=(S,\Sigma,E,\mu,\Phi,s_A)$ be an automaton. Then*

(i) $L^{\mathrm{ef}}(A)=\sum_{A_t\in \mathrm{Fin}_{\mathrm{ef}}(A)} L_{s_A,t}(A)L^{\mathrm{ef}}(A_t)$,

(ii) $L^{\mathrm{lf}}(A)=\sum_{A_t\in \mathrm{Fin}_{\mathrm{lf}}(A)} L_{s_A,t}(A)L^{\mathrm{lf}}(A_t)$,

(iii) $L^{\mathrm{pf}}(A)=\sum_{A_t\in \mathrm{Fin}_{\mathrm{pf}}(A)} L_{s_A,t}(A)L^{\mathrm{pf}}(A_t)$,

(iv) $L^{\mathrm{wf}}(A)=\sum_{A_t\in \mathrm{Fin}_{\mathrm{wf}}(A)} L_{s_A,t}(A)L^{\mathrm{wf}}(A_t)$

where the A_t are subautomata with an initial state t.

Proof. (i) ($\subseteq$): Let $w\in L^{\mathrm{ef}}(A)$; then $\exists p$: p ef-path s.t. $w=\Phi^{\lambda}(p)$. p final implies $\exists s_f$: s_f final state s.t. p starts in s_A and ends in s_f, whence $\exists A_{s_f}\in \mathrm{Fin}_{\mathrm{ef}}$: $w\in L_{s_A,s_f}(A)=L_{s_A,s_f}(A)L^{\mathrm{ef}}(A_{s_f})$ ($L^{\mathrm{ef}}(A_{s_f})=\{\lambda\}$).

If p is not final, then, by Definition 5.1 and Corollary 5.2, $\exists$subautomaton $A_t=(S^{\omega}(p),\Sigma,E^{\omega}(p),\mu',\Phi',t)$, and a suffix p' of p must have an 'edge-fair behavior' in A_t. Hence, $w\in L_{s_A,t}(A)L^{\mathrm{ef}}(A_t)$ for some $A_t\in \mathrm{Fin}_{\mathrm{ef}}$.

($\supseteq$): Obvious.

(ii), (iii), (iv): Analogously to (i). $\square$

6. Results on edge- and letter-fairness

The basic result of this section is that for a given automaton A the set of all ef-(lf-)words has a representation as an ω-regular expression. Thus we can embed the theory of edge- and letter-fairness in the quite reasonably understood theory of ω-regularity. All transformations made to get this ω-regular representation are constructive. As the equivalence problem is decidable for ω-automata, it is decidable whether two automata generate the same edge- or letter-fair language.

Lemma 6.1. *Let A be a final automaton. Then $L^{\mathrm{ef}}(A)\in \mathrm{Rat}^{\omega}_{\Sigma}$.*

Proof. Let $E=\{e_1,\ldots,e_n\}$ and $\mu(e)=({}^{\square}e,e^{\square})$. It is easy to see from the definition of edge-fairness that a path p is edge-fair within a final automaton A iff every edge of A occurs infinitely often in p. Thus we have

$$L^{\mathrm{ef}}(A)=(L_{s_A,{}^{\square}e_1}(A)\Phi^{\lambda}(e_1)L_{e_1^{\square},{}^{\square}e_2}(A)\Phi^{\lambda}(e_2)\ldots L_{e_{n-1}^{\square},{}^{\square}e_n}(A)\Phi^{\lambda}(e_n)L_{e_n^{\square},s_A}(A))^{\omega}.\quad\square$$

Lemma 6.2. *For every automaton A there exist an automaton B such that $L^{\mathrm{lf}}(A)=L^{\mathrm{ef}}(B)$*

Proof. Let $A=(S,\Sigma,E,\mu,\Phi,s_A)$ be an automaton. Construct B as follows: Transform every nontrivial letter-final subautomaton of A to an edge-final subautomaton of B by inserting ε-edges from the initial state of a letter-final subautomaton in A to an additional final subautomaton consisting of exactly this letter-final subautomaton.

As an example look at the transformation from Fig. 6 to Fig. 7. Thus, we define for $\mathrm{Fin}_{\mathrm{lf}}(A) = \{A^1, \ldots, A^n\}$ and $A^j = (S^j, \Sigma, E^j, \mu^j, \Phi^j, s_{i_j})$, $B := (S', \Sigma, E', \mu', \Phi', s_B)$ s.t.

$$\begin{aligned}
&s_B = s_A; \qquad \Phi' \text{ canonical};\\
&S' = S \cup \{s^j \mid s \in S^j, j = 1, \ldots, n\};\\
&E' = E \cup \{e^j \mid e \in E^j, j = 1, \ldots, n\} \cup \{\bar{e}^j \mid j = 1, \ldots, n\};\\
&\mu'(e) = \mu(e) \quad \text{for } e \in E;\\
&\mu'(e^j) = (s^j, t^j) \quad \text{where } \mu^j(e)(s, t);\\
&\mu'(\bar{e}_j) = (s_{i_j}, s^j_{i_j}), \quad j = 1, \ldots, n.
\end{aligned}$$

Now let $w \in L^{\mathrm{lf}}(A)$. If $w \in \Sigma^*$, then also $w \in L^{\mathrm{ef}}(B)$. Thus suppose $w \in \Sigma^\omega$. Then, $\exists p$: p lf-path s.t. $w = \Phi^\lambda(p)$. Hence, $\exists A^j \in \mathrm{Fin}_{\mathrm{lf}}(A)$: $w \in L_{s_A, s_{i_j}} L^{\mathrm{lf}}(A^j)$ (by Lemma 5.3). Because $E^j = E^\omega(p)$, $w \in L_{s_A, s_{i_j}} L^{\mathrm{ef}}(A'^j)$ holds, where A'^j is the representation of A^j as a final subautomaton in B. Hence, $w \in L^{\mathrm{ef}}(B)$ (as A'^j is final in B and by Lemma 5.3) and thus, $L^{\mathrm{lf}}(A) \subseteq L^{\mathrm{ef}}(B)$.

Let $w \in L^{\mathrm{ef}}(B)$. Then, $\exists a$ final subautomaton A' of B s.t. $w \in L_{s_A, s_{A'}} L^{\mathrm{ef}}(A')$, which implies $w \in L_{s_A, s_{A'}} L^{\mathrm{lf}}(A')$ (as every final subautomaton is also letter-final). Hence, $w \in L^{\mathrm{lf}}(A)$ (as $A' \in \mathrm{Fin}_{\mathrm{lf}}(A)$) and thus, $L^{\mathrm{ef}}(B) \subseteq L^{\mathrm{lf}}(A)$. □

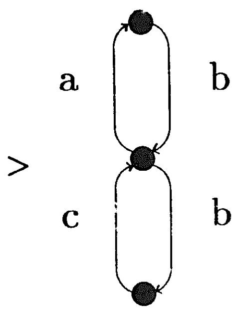


Fig. 6.

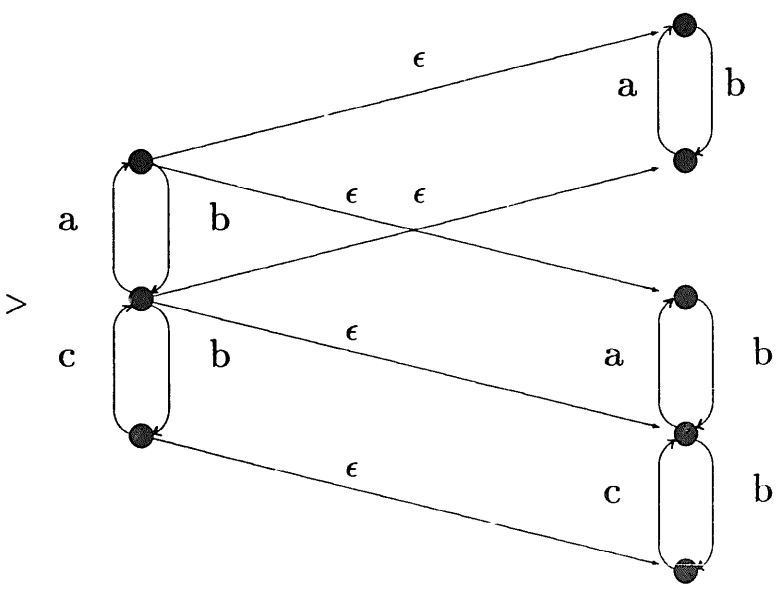


Fig. 7.

The reader may try to find a possible, more involved transformation that requires no additional ε-edges.

Applying the results of Büchi, Muller and Park [5, 17, 18] the following theorem holds.

Theorem 6.3. *For every automaton A there exist automata B_1, B_2, B_3 and sets F_1, F_2 such that*

(i) $L^{\mathrm{ef}}(A)=L^{\infty}(B_1, F_1)$,

(ii) $L^{\mathrm{lf}}(A)=L^{\infty}(B_2, F_2)$,

(iii) $L^{\mathrm{lf}}(A)=L^{\mathrm{ef}}(B_3)$.

The proof of Lemma 6.2 becomes constructive if we are able to present a procedure which finds all elements of $\mathrm{Fin}_{\mathrm{lf}}(A)$. This procedure is based on the following PPF-tree (Past, Present, Future) for letter-fairness.

Definition 6.4. Let $A=(S, \Sigma, E, \mu, \Phi, s_A)$ be an automaton. For $s_0 \in S$ the lf-PPF-tree is defined as $T=(N, R)$ s.t.

- $N \subseteq 2^{\Sigma} \times S \times 2^{\Sigma}$ is the set of nodes, and
- $R \subseteq N \times E \times N$ is the set of E-labeled arcs.

The root is $(\emptyset, s_0, \emptyset)$ and $(M, s, N) \to^{e} (M', s', N')$ iff $\mu(e)=(s, s')$ and $M'=M \cup \{\Phi(e)\}$ and

$$N'=(N \cup \{\Phi(e') \mid \exists t\colon \mu(e')=(s, t)\})-M'.$$

Example. An example is shown in Figs. 8(a) and 8(b), where, in Fig 8(b), is depicted the PPF-tree for s_A in Fig. 8(a) (identify edges with their names).

Lemma 6.5. *Let $A=(S, \Sigma, E, \mu, \Phi, s_A)$ be an automaton. Then $B=(S', \Sigma, E', \mu', \Phi', t) \in \mathrm{Fin}_{\mathrm{lf}}(A)$ iff*

(i) $\exists q$: *q path in the* lf-PPF-*tree of t*, $|q|<\infty$, $\Phi^{\lambda}(p) \neq \lambda$ s.t.

$$(\emptyset, t, \emptyset) \to^{q} (M, t, \emptyset) \quad \textit{for some } M \in S$$

and $S'=S^{\omega}(q^{\omega})$, $E'=E^{\omega}(q^{\omega})$, $\mu'=\mu_{/B}$, $\Phi'=\Phi_{/B}$ or

(ii) *B consists of a single sink state.*

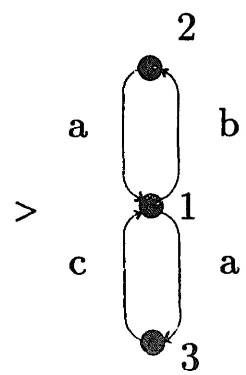


Fig. 8(a).

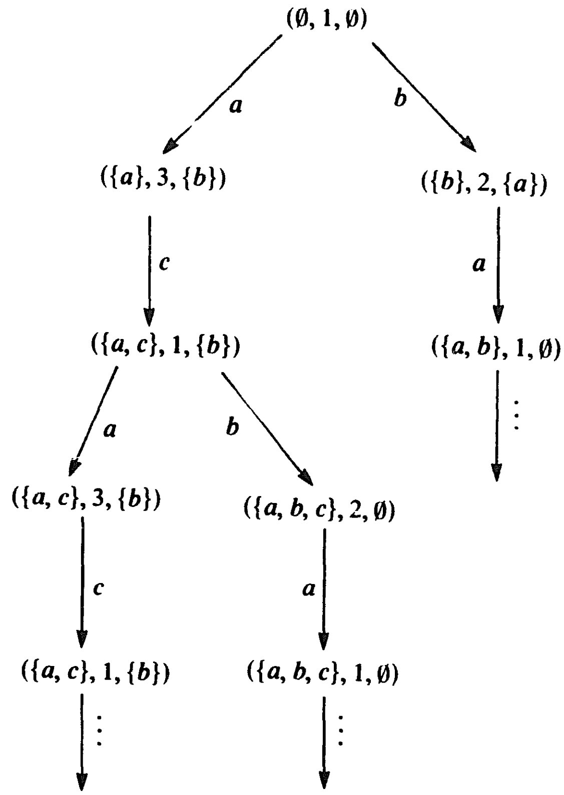


Fig. 8(b).

Proof. Let $B \in \mathrm{Fin}_{\mathrm{lf}}(A)$ (case (ii) is obvious). Then, $\exists p$: p lf-path in A with $|p| = \omega$ and $S' = S^\omega(p)$ and $E' = E^\omega(p)$ and $w = \Phi^\lambda(p)$.

Let $t \in S^\omega(p)$. Then, $\exists q \in E^*$: q is a infix of p; $t \to^q t$ and $\forall e \in E$; $e \in q$ (which implies, $S' = S^\omega(q^\omega)$ and $E' = E^\omega(q^\omega)$).

Assume $(\emptyset, t, \emptyset) \to^q (M, t, N)$ and $N \neq \emptyset$. Then, $\exists e \in E$ $\exists q', q'' \in E'^*$: $q = q'q''$, $t \to^{q'} t' \to^{q''} t$, $t' \to^{e}$ and $\Phi(e) \notin \Phi(q)$. Hence, $\exists t' \in S^\omega(p)$: e is an edge from t, but $\Phi(e)$ occurs not infinitely often in $\Phi(p)$ (as q consists of all edges of E'). Hence, p is not letter-fair (which is a contradiction) and thus, $N = \emptyset$ which implies (i).

For the second inclusion, consider the infinite path $p := q^\omega$. This path defines a subautomaton $B = (S^\infty(p), \Sigma, E^\infty(p), \mu', \Phi', t)$ and, with the same argument as above, it follows from the PPF-tree of t that p is a letter-fair path in A. Thus, $B \in \mathrm{Fin}_{\mathrm{lf}}(A)$. □

Every path like the one in Lemma 6.5(i) in such a PPF-tree defines a letter-final subautomaton. Thus we can find all (nontrivial) elements of $\mathrm{Fin}_{\mathrm{lf}}(A)$ by constructing the PPF-trees for all states $s \in S$. With a pumping-lemma like argument, it is sufficient to consider only finite trees given by a simple DFS algorithm from an inquired automaton A.

The proof of Theorem 6.3 is thus constructive. For (i) we apply Lemma 6.1. It thus remains to find all final subautomata. Therefore, we only have to find all connected subgraphs of a given graph. Applying a result of Tarjan [24] this can be done in linear time. As a consequence, the theory of edge-fairness can be trivially embedded in linear time into the quite reasonably understood theory of ω-regularity.

For (ii) and (iii) we have to find all elements of $\mathrm{Fin}_{\mathrm{lf}}(A)$, which can be done with the PPF-trees. Unfortunately, this procedure requires exponential time s.t. we still can embed the theory of letter-fairness into the theory of ω-regularity—but with an exponential transformation.

7. Algebraical treatment of path- and word-fairness

Word- and path-fairness have been introduced via automata. Another approach follows Kleene's regular expressions.

Definition 7.1. Let M be a language with $M \neq M^*$. Then $M^{\mathrm{pf}} := M$, $M^{\mathrm{wf}} := M$ with $\{\lambda\}^{\mathrm{pf}} = \{\lambda\}^{\mathrm{wf}} = \{\lambda\}$.

Let M be a language with $\{\lambda\} \neq M = M^*$. Then,

(i) $M^{\mathrm{wf}} := \{w \in \mathrm{adh}(M) \mid \forall u \in M\colon u \subseteq w\}$

(ii) $M^{\mathrm{pf}} := \{w \in M^{\omega} \mid \exists (w_i)_{i \in \mathbb{N}}. \forall i\colon\ (w_i < w_{i+1}$ and $w_i \in (M - \{\lambda\})^i)$ and $w = \lim_{i \to \infty} w_i$ and $\forall u \in M\ \exists k, l \in \mathbb{N}\colon w_k = w_l u\}$.

Corollary 7.2. *Let M be a language with $M = M^*$, $w \in M^{\mathrm{pf}}$, $w = \lim_{i \to \infty} w_i$ as in Definition* 7.1(ii) *and $u \in M$.*

(i) $\forall i \in \mathbb{N}\ \exists l_i, k_i\colon w_{k_i} = w_{l_i} u^i$.

(ii) $\forall k\ \forall i\ \exists r, l, m\colon m > k$ and $w_m = w_l u^{ir}$.

(iii) $M^{\mathrm{pf}} = \{w \in \Sigma^{\omega} \mid \exists (w_i)_{i \in \mathbb{N}}\colon w = \lim_{i \to \infty} w_i$ *and* $\forall u \in M\ \exists^{\omega} k, l\colon w_k = w_l u\}$.

Proof. (i): Obvious.

(ii): Proof by contradiction.

(iii): A consequence of (ii) by renaming the w_i-sequence. □

Definition 7.3. Rat_{Σ}, the set of all languages defined through a regular expression over Σ, is defined as usual:

(i) $0 \in \mathrm{Rat}_{\Sigma}$, $\forall a \in \Sigma\colon a \in \mathrm{Rat}_{\Sigma}$;

(ii) $\forall x, y \in \mathrm{Rat}_{\Sigma}\colon (x+y), (xy), x^* \in \mathrm{Rat}_{\Sigma}$.

$\mathrm{Rat}^{o}_{\Sigma}$ for $^{o} \in \{\mathrm{pf}, \mathrm{wf}\}$ is defined analogously:

(i) $\mathrm{Rat}_{\Sigma} \subseteq \mathrm{Rat}^{o}_{\Sigma}$;

(ii) $\forall y, z \in \mathrm{Rat}^{o}_{\Sigma}\ \forall x \in \mathrm{Rat}_{\Sigma}\colon (y+z), (xy), x^{o} \in \mathrm{Rat}^{o}_{\Sigma}$.

Note that $\mathrm{Rat}^{o}_{\Sigma}$ are classes of ∞-languages for $^{o} \in \{\mathrm{pf}, \mathrm{wf}\}$. As $\mathrm{Rat}^{\omega}_{\Sigma}$ will consist of ω-languages it only needs a slightly different definition:

$$\forall x, y \in \mathrm{Rat}_{\Sigma}\colon xy^{\omega} \in \mathrm{Rat}^{\omega}_{\Sigma}, \qquad \forall x, y \in \mathrm{Rat}^{\omega}_{\Sigma}\colon (x+y) \in \mathrm{Rat}^{\omega}_{\Sigma}.$$

We define an *interpretation* I as a mapping from expressions onto languages as usual:

$$I(0) = \emptyset, \quad I(a) = \{a\}, \quad I(x+y) = I(x) \cup I(y), \quad I(xy) = I(x)I(y),$$
$$I(x^*) = (I(x))^*, \quad I(x^{\omega}) = (I(x))^{\omega}, \quad I(x^{\mathrm{pf}}) = (I(x))^{\mathrm{pf}}, \quad I(x^{\mathrm{wf}}) = (I(x))^{\mathrm{wf}}.$$

We thus identify an expression with a language and say that M is a Rat_Σ^o-language or $M \in \mathrm{Rat}_\Sigma^o$ for $^o \in \{\omega, \mathrm{pf}, \mathrm{wf}\}$ iff there exists an expression $x \in \mathrm{Rat}_\Sigma^o$ with $M = I(x)$.

Lemma 7.4. (i) *For every final automaton* $A := (S, \Sigma, E, \mu, \Phi, s_A)$ *it holds that*

$$L^{\mathrm{pf}}(A) = (L_{s_A,s_A}(A))^{\mathrm{pf}}.$$

(ii) *For every final automaton* $A := (S, \Sigma, E, \mu, \Phi, s_A)$ *it holds that*

$$L^{\mathrm{wf}}(A) = h((L^{\varepsilon}_{s_A,s_A}(A))^{\mathrm{wf}})$$

where $L^{\varepsilon}_{s_A,s_A} = \{w \mid \exists p\colon p \text{ path from } s_A \text{ s.t. } w = \Phi(p)\}$ *(this is a language over* Σ_ε*) and* $h : \Sigma_\varepsilon^\infty \to \Sigma^\infty$ *is defined by* $h(\varepsilon) = \lambda$ *and* $h = \mathrm{id}$ *otherwise.*

Proof. (i) ($\subseteq$): $w \in L^{\mathrm{pf}}(A)$ implies $w \in \Sigma^\omega$, as there exists no final path p in A.

Let $p \in E^\omega$ be a path-fair path in A with $\Phi^\lambda(p) = w$. Because $s_A \in S^\omega(p)$ and as there exists an edge e s.t. $\Phi(e) \neq \varepsilon$, there exists an infinite sequence of paths $(p^i)_{i \in \mathbb{N}}$ with $p = p^1 p^2 \ldots$ s.t. *both* p^i is a path from s_A to s_A and $\forall i$: $\Phi^\lambda(p^i) \in L_{s_A,s_A}(A) - \{\lambda\}$, *and* p^i is in a certain sense 'minimal' which means:

$$\forall p_{i_1}, p_{i_2}, \text{ paths from } s_A \text{ to } s_A \text{ s.t. } \Phi(p_{i_1}) \neq \varepsilon^* \neq \emptyset(p_{i_2})\colon \quad p^i \neq p_{i_1} p_{i_2}.$$

Now, define $p_i := = p^1 p^2 \ldots p^i$, $w_i := \Phi^\lambda(p_i)$. It remains to show that

$$\forall u \in L_{s_A,s_A}(A)\colon \exists k, l \in \mathbb{N}\colon \quad w_k = w_l u.$$

Let $u \in L_{s_A,s_A}(A)$. There exists a path q from s_A to s_A s.t. $u = \Phi^\lambda(q)$ and q must be a subpath of p as p is pf! Thus, there exist $k, l \in \mathbb{N}$ with $p_k = p_l q' q q''$, as all p^i have been chosen minimal, with $q', q'' \in E^*$ and $\Phi(q')$, $\Phi(q'') \in \varepsilon^*$. Thus $w_k = \Phi^\lambda(p_k) = \Phi^\lambda(p_l q' q q'') = w_l u$.

($\supseteq$): $w \in (L_{s_A,s_A}(A))^{\mathrm{pf}}$. Hence, $\forall i \in \mathbb{N}\, \exists w_i \in \Sigma^*$: $w_i \in (L_{s_A,s_A}(A) - \{\lambda\})^i$ and $w = \lim_{i\to\infty} w_i$. So $\forall j \in \mathbb{N}\, \exists w^j \in (L_{s_A,s_A}(A) - \{\lambda\})$: $\forall i \in \mathbb{N}$: $w_i = w^1 \ldots w^i$ and thus, $\forall j \in \mathbb{N}\, \exists p^j$ a nonempty path from s_A to s_A: $\Phi^\lambda(p^j) = w^j$.

Now, define $\forall i \in \mathbb{N}$: $p_i := p^1 \ldots p^i$ and $p := \lim_{i\to\infty} p_i$. Obviously, $\Phi^\lambda(p) = w$.

It remains to prove that p is path-fair or that there exists a path-fair path $\bar{p}$ s.t. $\Phi^\lambda(\bar{p}) = w$.

Let $s \in S^\omega(p)$ and q be a finite path starting in s. Because A is final, there exist paths q', q'' s.t. q' starts in s_A and ends in s, and q'' starts in s and ends in s_A. Thus, $\bar{q} := q' q q''$ is a path from s_A to s_A with $u := \Phi^\lambda(\bar{q}) \in L_{s_A,s_A}(A)$.

Therefore, consider the countable set

$$P = \{q \mid q \text{ finite path from } s_A \text{ to } s_A\} =: \{q_1, q_2, \ldots\}$$

and define $u_i := \Phi^\lambda(q_i)$. There exists a sequence $(r_i, k_i, l_i) \in \mathbb{N}^3$ s.t. $w_{k_i} = w_{l_i} u^{r_i}$ and $l_i > k_{i-1}$ (by Corollary 7.2 if $u_i \neq \lambda$ and obviously if $u_i = \lambda$).

Now let

$$p_i := p_{i-1} p'_{i-1} q_i^{r_i},$$

where p'_{i-1} is determined by $\Phi^\lambda(p_{i-1} p'_{i-1}) = w_{l_i}$. Then $p := \lim_{i\to\infty} p_i$ is a well-defined infinite path with $\Phi^\lambda(p) = w$ and from the choice of q_i it follows that p is path-fair.

(ii) ($\supseteq$): $w \in (L^{\varepsilon}_{s_A,s_A}(A))^{\mathrm{wf}}$ whence $w \in \overline{\mathrm{Pref}(L^{\varepsilon}_{s_A,s_A}(A))} = \overline{\mathrm{Pref}(L^{\varepsilon}_{s_A,s_A}(A)^*)}$. This implies $\forall i \in \mathbb{N}\ \exists w_i \in \mathrm{Pref}(L^{\varepsilon}_{s_A,s_A}(A)^*)$: $w = \lim_{i\to\infty} w_i$ and thus, $\forall i \in \mathbb{N}\ \exists p_i$ paths in A: $w_i = \Phi(p_i)$ and $p_i < p_{i+1}$.

For $p := \lim_{i\to\infty} p_i$ it holds that $\Phi(p) = w$. We have to show that p is word-fair: Let $s \in S^{\omega}(p)$, q be a finite path starting in s, p_0 be a path from s_A to s with $p_0 < p$. Then $u := \Phi(p_0 q) \in \mathrm{Pref}(L^{\varepsilon}_{s_A,s_A}(A)^*)$ and hence, $\exists u'$: $uu' \in L^{\varepsilon}_{s_A,s_A}(A)$ which implies $uu' \subseteq w \Rightarrow \Phi(q) \subseteq uu' \subseteq \Phi(p)$.

($\subseteq$): $w \in L^{\mathrm{wf}}(A)$ whence $\exists p$ word-fair path: $w = \Phi^{\lambda}(p)$. Let $\bar{w} := \Phi(p)$; thus, $w = h(\bar{w})$. Let $p_i < p$ s.t. $p = \lim_{i\to\infty} p_i$ and $\bar{w}_i := \Phi(p_i) \in \mathrm{Pref}(L^{\varepsilon}_{s_A,s_A}(A))$, $\bar{w} = \lim_{i\to\infty} \bar{w}_i$.

We have to prove

$$\forall u \in L^{\varepsilon}_{s_A,s_A}(A)\colon u \subseteq \bar{w}.$$

Let $u \in L^{\varepsilon}_{s_A,s_A}(A)$; p infinite implies $S^{\omega}(p) \neq \emptyset$, whence $\forall s \in S^{\omega}(p)\ \exists q$ path from s to s_A. Let q_u be a path starting from s_A s.t. $\Phi(q_u) = u$. Then, qq_u is a path in A starting from $s \in S^{\omega}(p)$ and thus, $\Phi(qq_u) \subseteq \bar{w}$ (as p is word-fair) and so $u \subseteq \bar{w}$. □

With this lemma and Lemma 5.3 one proves a canonical representation for fair languages.

Theorem 7.5 (Algebraical Normal-form Theorem). *For every automaton A there exist finite sets I, $J \subseteq \mathbb{N}$, automata B_i, B'_j, final subautomata C_i and D_j s.t.*

(i) $L^{\mathrm{pf}}(A) = \sum_{i\in I} L(B_i)(L_{s_{C_i},s_{C_i}}(C_i))^{\mathrm{pf}} + L(B_0)$;

(ii) $L^{\mathrm{wf}}(A) = h(\sum_{j\in J} L(B'_j)(L^{\varepsilon}_{s_{D_j},s_{D_j}}(D_j))^{\mathrm{wf}} + L(B'_0))$.

This immediately implies our main result.

Theorem 7.6 (Main Theorem)

(i) $\mathrm{Rec}^{\mathrm{pf}}_{\Sigma} = \mathrm{Rat}^{\mathrm{pf}}_{\Sigma}$.

(ii) $\mathrm{Rec}^{\mathrm{wf}}_{\Sigma} = h(\mathrm{Rat}^{\mathrm{wf}}_{\Sigma_\varepsilon})$.

It should be noted that we have not stated an algebraical counterpart "$\mathrm{Rat}^{\mathrm{ef}}_{\Sigma}$" and "$\mathrm{Rat}^{\mathrm{lf}}_{\Sigma}$". In fact we do not have such a definition.

We would also like to mention that

$$\mathrm{Rat}^{\mathrm{wf}}_{\Sigma} = \mathrm{Rec}^{\mathrm{wf}}_{\Sigma}$$

holds. However, this proof is more involved and requires some techniques that will be presented in a second paper.

Lemma 7.7. (i) $\Sigma^{\omega} \notin \mathrm{Rec}^{\mathrm{pf}}_{\Sigma}$ *and* $\Sigma^{\omega} \notin \mathrm{Rec}^{\mathrm{wf}}_{\Sigma}$ *for* $|\Sigma| > 1$.

(ii) $\mathrm{Rec}^{\mathrm{pf}}_{\Sigma}$ *and* $\mathrm{Rec}^{\mathrm{wf}}_{\Sigma}$ *are not closed under complement, substitution and inverse homomorphism.*

(iii) $\mathrm{Rat}^{\mathrm{pf}}_{\Sigma}$ *is closed under homomorphism.*

(iv) $\mathrm{Rat}^{\mathrm{wf}}_{\Sigma}$ *is closed under* λ*-free homomorphism.*

Proof. (i) (*Sketch of proof*): Assume $\Sigma^\omega \in \mathrm{Rec}_\Sigma^{\mathrm{pf}}$. Then, there exists an automaton A s.t. $L^{\mathrm{pf}}(A) = \Sigma^\omega$. Hence, $|\mathrm{Fin}_{\mathrm{pf}}(A)| = \omega$. The details are left to the reader. The same argument leads to a proof for $\Sigma^\omega \notin \mathrm{Rec}_\Sigma^{\mathrm{wf}}$.

(ii): Assume $\mathrm{Rec}_\Sigma^{\mathrm{xf}}$ is closed under complement. ($xf \in \{\mathrm{pf}, \mathrm{wf}\}$). As $\Sigma^* \in \mathrm{Rec}_\Sigma^{\mathrm{xf}}$, consider the complement of Σ^* within Σ^∞. But Σ^ω is no element of either $\mathrm{Rec}_\Sigma^{\mathrm{pf}}$ or $\mathrm{Rec}_\Sigma^{\mathrm{wf}}$.

Assume $\mathrm{Rec}_\Sigma^{\mathrm{xf}}$ is closed under substitution. Consider the special substitution $a \to \Sigma$. As $a^\omega \in \mathrm{Rec}_\Sigma^{\mathrm{xf}}$, it follows that $\Sigma^\omega \in \mathrm{Rec}_\Sigma^{\mathrm{xf}}$.

Assume $\mathrm{Rec}_\Sigma^{\mathrm{xf}}$ is closed under inverse homomorphism. Consider $h: \Sigma \to \Sigma: x \to a$ $\forall x \in \Sigma$. Then, $h^{-1}(a^\omega) = \Sigma^\omega \in \mathrm{Rec}_\Sigma^{\mathrm{xf}}$. Here substitution and homomorphism are the canonical extensions to Σ^ω.

(iii): Let $M \in \mathrm{Rec}_\Sigma^{\mathrm{pf}}$, $f: \Sigma^* \to \Gamma^*$ be a homomorphism defined by $f(a) = w_a \in \Gamma^*$ $\forall a \in \Sigma$. Let $M = L^{\mathrm{pf}}(A)$. Replace every edge labeled by a in A by a path of length $|w_a|$ labeled with w_a for $|w_a| > 0$. If $|w_a| = 0$, replace this edge by an edge labeled with ε. Let A' denote this new automaton. Thus, $f(M) = L^{\mathrm{pf}}(A')$ obviously holds as every pf path p in A that uses some edge e now has to use a corresponding path q_e in A' where $\Phi^\lambda(q_e) = \Phi^\lambda(e)$.

(iv): For word-fairness a further argument is needed in addition to the one in (iii). Let $M \in f(L^{\mathrm{wf}}(A))$, $f: \Sigma^* \to \Gamma^*$ be a homomorphism defined by $f(a) = w_a \in \Gamma^+$ $\forall a \in \Sigma$ and let $m := \max_{a \in \Sigma} |w_a|$. We use a coding $c: \Gamma^* \to \Gamma_e^*$ s.t. $h(c(w)) = h(w)$ (for h, see Lemma 7.4) $\forall w \in \Gamma^*$ and s.t. $c \circ f: \Sigma^* \to \Gamma_\varepsilon^*$ is a mapping s.t. $c(f(a)) = \varepsilon^{m+1-|w_a|} w_a$.

By Theorem 7.5 we may assume that A is word-final (the general case is similar). Again, replace every edge labeled with a in A by a path q labeled with $c \circ f(a)$ to receive A'. Thus, A' is also word-final and every wf path p in A corresponds to a wf path $\bar{p}$ in A' with $f(\Phi^\lambda(\bar{p}))$. Thus, $f(M) = h(L^{\mathrm{wf}}(A'))$. □

As we will prove $\mathrm{Rec}_\Sigma^{\mathrm{pf}} = \mathrm{Rec}_\Sigma^{\mathrm{wf}}$ in a second paper, we will then have proofs for some more closure properties.

We now turn our interest to some algebraical properties of path- and word-fair languages.

Lemma 7.8. *Let M and N be $*$-languages over Σ with $M = M^*$ and $N = N^*$.*

(i) $M^{\mathrm{pf}} = MM^{\mathrm{pf}} = M^* M^{\mathrm{pf}}$.

(ii) $M^{\mathrm{pf}} \subseteq N^{\mathrm{pf}} \not\Rightarrow M^\omega \subseteq N^\omega$.

(iii) $M \subseteq N \not\Rightarrow M^{\mathrm{pf}} \subseteq N^{\mathrm{pf}}$; $M^\omega \subseteq N^\omega \not\Rightarrow M^{\mathrm{pf}} \subseteq N^{\mathrm{pf}}$.

(iv) $M^{\mathrm{pf}} \subseteq N^{\mathrm{pf}} \Rightarrow M \subseteq \mathrm{Pref}(N)$.

(v) $\mathrm{Pref}(M) = \mathrm{Pref}(N) \Rightarrow M^{\mathrm{pf}} \cap N^{\mathrm{pf}} \neq \emptyset$.

Proof. (i): This is obvious because of Corollary 7.2.

(ii): $M = (a+b)^*$, $N = (a^*b)^*$; hence $M^{\mathrm{pf}} = N^{\mathrm{pf}}$ but $a^\omega \in M^\omega - N^\omega$.

(iii): $M = a^*$, $N = (a+b)^*$. Hence, $M \subseteq N$ and $M^\omega \subseteq N^\omega$, but $M^{\mathrm{pf}} = a^\omega \not\subseteq N^{\mathrm{pf}}$ as $w \in N^{\mathrm{pf}}$ implies $b \in w$.

(iv): $\forall u \in M\ \exists w \in M^{\mathrm{pf}}$: $u < w$. Then,

$$M^{\mathrm{pf}} \subseteq N^{\mathrm{pf}} \Rightarrow w \in N^{\mathrm{pf}} \Rightarrow \exists (w_i)_{i\in\mathbb{N}} w_i \in (N-\{\lambda\})^i \text{ and } w = \lim_{i\to\infty} w_i.$$

Hence, $\exists k$: $u < w_k$ and thus, $u \in \mathrm{Pref}(N^*) = \mathrm{Pref}(N)$.

(v): Construct $w \in M^{\mathrm{pf}} \cap N^{\mathrm{pf}}$ as follows: Let $(x_i)_{i\in\mathbb{N}}$ be an enumeration of M^* and $(y_i)_{i\in\mathbb{N}}$ an enumeration of N^*, and $M^* = M \subseteq \mathrm{Pref}(M) = \mathrm{Pref}(N)$. Then,

$$\forall u \in M\ \exists v \in N: \quad u < v.$$

Set $w^1 := u$, $w^2 := x_1 \in M$, $\bar{w}^1 := v$, $\bar{w}^2 := y_1 \in N$. Let $w^1, \ldots, w^{2n} \in M$ and $\bar{w}^1, \ldots, \bar{w}^{2n}$ be defined inductively s.t.

$$w^1 \ldots w^{2n} < \bar{w}^1 \ldots \bar{w}^{2n} \in N^* = N \subseteq \mathrm{Pref}(N) = \mathrm{Pref}(M)$$

where the w^i and $\bar{w}^i$ are defined as follows:

$$\exists u \in M: \quad \bar{w}^1 \ldots \bar{w}^{2n} < w^1 \ldots w^{2n} u,$$

$$\exists v \in N: \quad w^1 \ldots w^{2n} u x_{n+1} < \bar{w}^1 \ldots \bar{w}^{2n} v.$$

Set $w^{2n+1} := u$, $w^{2(n+1)} := x_{n+1} \in M$ and $\bar{w}^{2n+1} := v$, $\bar{w}^{2(n+1)} := y_{n+1} \in N$. Now, define

$$w_i := w^1 \ldots w^i \quad \text{and} \quad \bar{w}_i := \bar{w}^1 \ldots \bar{w}^i;$$

then it obviously holds that

$$\lim_{i\to\infty} w_i = \lim_{i\to\infty} \bar{w}_i, \qquad \lim_{i\to\infty} w_i \in M^{\mathrm{pf}}, \qquad \lim_{i\to\infty} \bar{w}_i \in N^{\mathrm{pf}}. \qquad \square$$

Lemma 7.9. *Let M and N be $*$-languages over Σ with $M = M^*$, $N = N^*$.*

(i) $M \subseteq N \not\Rightarrow M^{\mathrm{wf}} \subseteq N^{\mathrm{wf}}$ *and* $M^{\omega} \subseteq N^{\omega} \not\Rightarrow M^{\mathrm{wf}} \subseteqq N^{\mathrm{wf}}$.

(ii) $M^{\mathrm{wf}} = N^{\mathrm{wf}} \Leftrightarrow \mathrm{adh}(M) = \mathrm{adh}(N) \Leftrightarrow \mathrm{Pref}(M) = \mathrm{Pref}(N)$.

If M and N are also regular it holds

(iii) $M^{\mathrm{wf}} \cap N^{\mathrm{wf}} \neq \emptyset \Rightarrow \mathrm{Suf}(M^{\mathrm{wf}}) = \mathrm{Suf}(N^{\mathrm{wf}}) \Leftrightarrow \mathrm{Inf}(M) = \mathrm{Inf}(N)$ *where* $\mathrm{Suf}(L) = \{w \mid \exists u \in \Sigma^*: uw \in L\}$ *is the set of suffixes of L and* $\mathrm{Inf}(L) = \{w \mid \exists u, v \in \Sigma^*: uvw \in L\}$ *is the set of* infixes of L.

Proof. (i): Use the same example as for part (iii) in the previous lemma.

(ii): Suppose $M^{\mathrm{wf}} = N^{\mathrm{wf}}$ and let $w \in \mathrm{adh}(M) = \mathrm{adh}(M^*)$. $\exists (w_i)_{i\in\mathbb{N}}$: $w_i \in \mathrm{Pref}(M^*)$, $w = \lim_{i\to\infty} w_i$. Then $\forall i \in \mathbb{N}\ \exists u_i \in \Sigma^{\omega}$: $w_i u_i \in M^{\mathrm{wf}} = N^{\mathrm{wf}}$ which implies $\forall i \in \mathbb{N}$: $w_i \in \mathrm{Pref}(N^*)$. Hence, $\lim_{i\to\infty} w_i \in \mathrm{adh}(N)$ and thus, $\mathrm{adh}(M) \subseteq \mathrm{adh}(N)$ and, by symmetry, $\mathrm{adh}(M) = \mathrm{adh}(N)$.

Suppose $\mathrm{adh}(M) = \mathrm{adh}(N)$ and let $w \in \mathrm{Pref}(M)$. Then, $\exists u \in \Sigma^{\omega}$: $wu \in \mathrm{adh}(M) = \mathrm{adh}(N) \Rightarrow w \in \mathrm{Pref}(N)$. By symmetry again, it holds that $\mathrm{Pref}(M) = \mathrm{Pref}(N)$.

Suppose $\mathrm{Pref}(M) = \mathrm{Pref}(N)$ and let $w \in M^{\mathrm{wf}}$. Then, $\exists (w_i)_{i\in\mathbb{N}} \forall i \in \mathbb{N}\ w_i \in \mathrm{Pref}(M^*)$: $w = \lim_{i\to\infty} w_i$ and $\forall u \in M$: $u \subseteq w$. Let $v \in N \subseteq \mathrm{Pref}(N) = \mathrm{Pref}(M)$. Then $\exists v' \in \Sigma^*$: $vv' \in M \Rightarrow vv' \subseteq w \Rightarrow w \in N^{\mathrm{wf}}$ as $w \in \mathrm{adh}(N)$. And again by symmetry: $M^{\mathrm{wf}} = N^{\mathrm{wf}}$.

(iii): Suppose $u \in M^{\text{wf}} \cap N^{\text{wf}}$ and let $w \in \text{Suf}(M^{\text{wf}})$. Then, $u = \lim_{i\to\infty} u_i = \lim_{i\to\infty} \bar{u}_i$ where $u_i \in (M-\{\lambda\})^i$, $\bar{u}_i \in (N-\{\lambda\})^i$. $\exists v \in \Sigma^*$: $vw \in M^{\text{wf}}$, $vw = \lim_{i\to\infty} vw_i$ and $\forall i$: $vw_i \in \text{Pref}(M)$. Hence,

$$\begin{aligned}\exists z_i\colon vw_iz_i \in M &\Rightarrow \forall i \in \mathbb{N}\colon vw_iz_i' \subseteq u \quad (\text{as } u \text{ is word-fair})\\ &\Rightarrow \forall i \in \mathbb{N}\ \exists j \in \mathbb{N}\colon vw_i \subseteq u_j\\ &\Rightarrow \forall i \in \mathbb{N}\ \exists j \in \mathbb{N}\ \exists z_i \in \Sigma^*\colon z_ivw_i < u_j\\ &\Rightarrow \forall i \in \mathbb{N}\ \exists z_i \in \Sigma^*\colon vw_i \in {}_{z_i}\text{Pref}(N),\end{aligned}$$

where ${}_z\text{Pref}(N) := \{x \mid zx \in \text{Pref}(N)\}$. As N is regular, it follows that $\text{Pref}(N)$ and ${}_{z_i}\text{Pref}(N)$ are regular and $D(\text{Pref}(N)) := \{{}_{z_i}\text{Pref}(N) \mid z_i \in \Sigma^*\}$ is finite.

$$\begin{aligned}&\Rightarrow \exists k \in \mathbb{N}\colon D(\text{Pref}(N)) = \{{}_{x_0}\text{Pref}(N), \ldots, {}_{x_k}\text{Pref}(N)\}\\ &\Rightarrow \exists (i_v)_{v\in\mathbb{N}}\colon \forall v\colon vw_{i_v} \in {}_x\text{Pref}(N) \text{ for some } x \in \{x_0, \ldots, x_k\}\colon\\ &\qquad w = \lim_{v\to\infty} w_{i_v}\\ &\Rightarrow xvw \in \text{adh}(N).\end{aligned}$$

Let $y \in N$; then $y \subseteq u$.

$$u = \lim_{i\to\infty} \bar{u}_i,\ \bar{u}_i \in \text{Pref}(N) \Rightarrow \exists i_0\colon y \subseteq \bar{u}_{i_0}.$$

$$u = \lim_{i\to\infty} u_i,\ u_i \in \text{Pref}(M) \Rightarrow \exists j_0\colon y \subseteq u_{j_0}.$$

Hence, $\exists z \in M$: $y \subseteq z \subseteq vw \in M^{\text{wf}}$ from which it follows that $y \subseteq xvw \in \text{adh}(N)$. Thus $xvw \in N^{\text{wf}}$. Hence, $xvw \in N^{\text{wf}}$ and $w \in \text{Suf}(N^{\text{wf}})$. Thus $\text{Suf}(M^{\text{wf}}) \subseteq \text{Suf}(N^{\text{wf}})$. Again, by symmetry, $\text{Suf}(N^{\text{wf}}) = \text{Suf}(M^{\text{wf}})$.

Now, suppose $\text{Suf}(M^{\text{wf}}) = \text{Suf}(N^{\text{wf}})$ and $w \in \text{Inf}(M)$. Then, $\exists z \in \Sigma^*$, $y \in \Sigma^\omega$: $zwy \in M^{\text{wf}}$ implying $wy \in \text{Suf}(M^{\text{wf}}) = \text{Suf}(N^{\text{wf}})$, $zwy = \lim_{i\to\infty} zwy_i$, $zwy_i \in \text{Pref}(M)$ and $zwy = \lim_{i\to\infty} zw\bar{y}_i$, $zw\bar{y}_i \in \text{Pref}(N)$. Thus, $w \in \text{Inf}(N)$ and, by symmetry, $\text{Inf}(M) = \text{Inf}(N)$.

At last, suppose $\text{Inf}(M) = \text{Inf}(N)$ and let $w \in \text{Suf}(M^{\text{wf}})$. Then, $\exists x \in \Sigma^*$: $xw \in M^{\text{wf}}$ implying $xw = \lim_{i\to\infty} xw_i$ and $xw_i \in \text{Pref}(M) \subseteq \text{Inf}(M) = \text{Inf}(N)$. Hence,

$$\forall i \in \mathbb{N}\ \exists z_i \in \Sigma^*\colon \quad z_ixw_i \in \text{Pref}(N)$$

and

$$\forall i \in \mathbb{N}\ \exists z_i \in \Sigma^*\colon \quad xw_i \in {}_{z_i}\text{Pref}(N)$$

As before, we find an infinite sequence $(i_\nu)_{\nu\in\mathbb{N}}$ and some $z \in \Sigma^*$ s.t.

$$\forall \nu \in \mathbb{N}\colon \quad xw_{i_\nu} \in {}_z\text{Pref}(N) \quad \text{and} \quad \lim_{\nu\to\infty} zxw_{i_\nu} = zxw.$$

Hence, $zxw \in \text{adh}(N)$ as before. Let $y \in N \subseteq \text{Inf}(N) = \text{Inf}(M)$. Then, $\exists u, v \in \Sigma^*$: $uyv \in M$ implying

$$uyv \subseteq w \Rightarrow y \subseteq w \Rightarrow y \subseteq zxw \Rightarrow zxw \in N^{\text{wf}} \Rightarrow w \in \text{Suf}(N^{\text{wf}})$$

and therefore, $\text{Suf}(M^{\text{wf}}) = \text{Suf}(N^{\text{wf}})$. $\square$

Lemma 7.10. *Let M, N and R be regular languages over Σ with $M=M^*$, $N=N^*$ and $R=R^*$.*

(i) $\mathrm{Inf}(M)=\mathrm{Inf}(N) \not\Rightarrow M^{\mathrm{wf}} \cap N^{\mathrm{wf}} \neq \emptyset$.

(ii) $M^{\mathrm{wf}} = N^{\mathrm{wf}} \cup R^{\mathrm{wf}}$ *and* $N^{\mathrm{wf}} \neq \emptyset \neq R^{\mathrm{wf}} \Rightarrow \mathrm{Inf}(M)=\mathrm{Inf}(N)=\mathrm{Inf}(R)$.

Proof. (i): Consider the languages $M=(ab)^*$ and $N=(ba)^*$. We have $M^{\mathrm{wf}} \cap N^{\mathrm{wf}} = \emptyset$, but $\mathrm{Inf}(M)=\mathrm{Inf}(N)$.

(ii): $M^{\mathrm{wf}} = N^{\mathrm{wf}} \cup R^{\mathrm{wf}}$ implies $M^{\mathrm{wf}} \cap N^{\mathrm{wf}} \neq \emptyset$ and $M^{\mathrm{wf}} \cap N^{\mathrm{wf}} \neq \emptyset$. Now, apply the previous lemma. □

These results are a hint that these fair languages are quite sparse within the topology of ω-languages. By Lemma 6.8(ii) we have a decision procedure for $M^{\mathrm{wf}} = N^{\mathrm{wf}}$ for regular *-languages M and N. However, this does not solve the fair-language equivalence problem.

8. Büchi- and Muller-automata

There may rise some criticism why we studied fairness for a new 'exotic' type of automaton instead of for standard automata for ω-languages like Muller- or Büchi-automata. However, all these approaches coincide in many important cases, as final states are not required for fairness.

Definition 8.1. A Büchi-automaton A is a 2-tuple $A=(A', F)$, where A' is an automaton and $F \subseteq S_{A'}$ is a set of final states.

A Muller-automaton A is a 2-tuple $A=(A', \Gamma)$, where A' is an automaton and $\Gamma \subseteq 2^{S_{A'}}$.

A path p is called Büchi-accepting (Muller-accepting) iff it is final or $S^{\omega}(p) \cap F \neq \emptyset$ ($S^{\omega}(p) \in \Gamma$).

With this preliminary definition we can easily define the fair languages of Büchi- and Muller-automata.

Definition 8.2. Let A be a Büchi-automaton. The x-fair language of A for every fairness notion x is defined as

$$L^{\text{B-}x\text{f}}(A) := \{w \in \Sigma^{\infty} \mid \exists \text{path } p\colon w=\Phi^{\lambda}(p) \text{ and } p \text{ is Büchi-accepting and } x\text{-fair}\}.$$

Analogously, the xf language of a Muller-automaton A can be defined as

$$L^{\text{M-}x\text{f}}(A) := \{w \in \Sigma^{\infty} \mid \exists \text{path } p\colon w=\Phi^{\lambda}(p) \text{ and } p \text{ is Muller-accepting and } x\text{-fair}\}.$$

Thereby the classes $\mathrm{Rec}_{\Sigma}^{\text{B-}x\text{f}}$ and $\mathrm{Rec}_{\Sigma}^{\text{M-}x\text{f}}$ of x-fair languages accepted by Büchi and Muller-automata are defined. Also one easily defines the classes $\mathrm{Rec}_{\Sigma,\varepsilon\text{-free}}^{\text{M/B-}x\text{f}}$ of languages defined by various automata without ε-edges. For technical reasons we add all languages of the type $M=M' \cup \{\lambda\}$ where $M' \in \mathrm{Rec}_{\Sigma,\varepsilon\text{-free}}^{\text{M/B-}x\text{f}}$ to these classes, as the empty word cannot be accepted in addition to a nonempty set without ε-edges and final states.

Note that $\mathrm{Rec}^{\text{M-pf}}_{\Sigma,\varepsilon\text{-free}}$ coincides with the class of strongly 3-accepted languages of Rosier and Yen [23]. Note also that classical Büchi- and Muller-automata operate without ε-edges. These cases are handled now too. We will prove in the following that no final states or ε-edges are required in a theory of path-fairness. This is an important difference to the theory of ω-regularity as it is well-known that, e.g., the language $(ab^*)^\omega$ cannot be recognized without final states. Thus, the difficulties arising in a proof of the Main Theorem for Büchi- and Muller-automata can be avoided in a theory of fairness.

Lemma 8.3. *For* $\mathrm{xf} \in \{\mathrm{pf}, \mathrm{ef}\}$ *it holds that*

(i) $\mathrm{Rec}^{\text{B-xf}}_{\Sigma} \subseteq \mathrm{Rec}^{\mathrm{xf}}_{\Sigma}$ *and* $\mathrm{Rec}^{\text{M-xf}}_{\Sigma} \subseteq \mathrm{Rec}^{\mathrm{xf}}_{\Sigma}$;

(ii) $\mathrm{Rec}^{\text{B-xf}}_{\Sigma,\varepsilon\text{-free}} \subseteq \mathrm{Rec}^{\mathrm{xf}}_{\Sigma,\varepsilon\text{-free}}$ *and* $\mathrm{Rec}^{\text{M-xf}}_{\Sigma,\varepsilon\text{-free}} \subseteq \mathrm{Rec}^{\mathrm{xf}}_{\Sigma,\varepsilon\text{-free}}$.

Proof. Let $M = L^{\text{B/M-xf}}(A)$ for a Büchi-automaton $A = (A', F)$ (Muller-automaton $A = (A', \Gamma)$ respectively). From Lemma 5.3 we get $L^{\mathrm{xf}}(A') = \sum_{i \in I} L(B_i) L^{\mathrm{xf}}(C_i)$ for a finite set I.

Let $D := \{C_i \mid i \in I \text{ and } S_{C_i} \cap F \neq \emptyset\}$ ($D := \{C_i \mid i \in I \text{ and } S_{C_i} \in \Gamma\}$ respectively). Now, delete all final subautomata not in D and all states that cannot be prolonged into some subautomaton in D. Let B be the resulting automaton. It obviously holds that $L^{\mathrm{xf}}(B) = L^{\text{B-xf}}(A)$ ($= L^{\text{M-xf}}(A)$ respectively).

If A contains no ε-edges, the same holds for B. □

Lemma 8.4. *For* $\mathrm{xf} \in \{\mathrm{pf}, \mathrm{ef}\}$ *it holds that*

(i) $\mathrm{Rec}^{\mathrm{xf}}_{\Sigma} \subseteq \mathrm{Rec}^{\text{B-xf}}_{\Sigma}$ *and* $\mathrm{Rec}^{\mathrm{xf}}_{\Sigma} \subseteq \mathrm{Rec}^{\text{M-xf}}_{\Sigma}$;

(ii) $\mathrm{Rec}^{\mathrm{xf}}_{\Sigma,\varepsilon\text{-free}} \subseteq \mathrm{Rec}^{\text{B-xf}}_{\Sigma,\varepsilon\text{-free}}$ *and* $\mathrm{Rec}^{\mathrm{xf}}_{\Sigma,\varepsilon\text{-free}} \subseteq \mathrm{Rec}^{\text{M-xf}}_{\Sigma,\varepsilon\text{-free}}$.

Proof. (i): Let A be an automaton. Define

$$F_A := \bigcup_{A_t \in \mathrm{Fin}_{\mathrm{xf}}(A)} S_{A_t}.$$

Then $B := (A, F_A)$ is a Büchi-automaton with $L^{\text{B-xf}}(B) = L^{\mathrm{xf}}(A)$. If A is ε-free, then so is B.

(ii): Very similarly to (i): define $\Gamma := \{S_{A_t} \mid A_t \in \mathrm{Fin}_{\mathrm{xf}}(A)\}$ and $B := (A, \Gamma)$. Again, $L^{\text{M-xf}}(B) = L^{\mathrm{xf}}(A)$ and B is ε-free if A has this property. □

Lemma 8.5. $\mathrm{Rec}^{\mathrm{pf}}_{\Sigma} = \mathrm{Rec}^{\mathrm{pf}}_{\Sigma,\varepsilon\text{-free}}$.

Proof. Eliminate the ε-edges by applying the algorithm of Albert and Ottmann [2, Theorem 1.11]. The last step is omitted, as we have no final states here. Additionally, if there is an edge from a state s to a sink state with label ε, this edge is replaced by new edges to this sink state. For every edge e leading to s, a new edge with the same label leading to the sink state is introduced. It is easily seen that this procedure will not change the path-fair languages. □

This algorithm cannot be applied to the other fairness notions. A similar lemma does hold for edge-fairness, but the proof is much more involved and is not presented here.

Summary 8.6. *For* $xf \in \{ef, pf\}$ *it holds that*

$$\mathrm{Rec}_{\Sigma}^{xf} = \mathrm{Rec}_{\Sigma,\varepsilon\text{-free}}^{xf} = \mathrm{Rec}_{\Sigma}^{B\text{-}xf} = \mathrm{Rec}_{\Sigma,\varepsilon\text{-free}}^{B\text{-}xf} = \mathrm{Rec}_{\Sigma}^{M\text{-}xf} = \mathrm{Rec}_{\Sigma,\varepsilon\text{-free}}^{M\text{-}xf}.$$

These relations do not hold for letter- or word-fairness in Büchi- or Muller-automata in this simple form. These connections will be treated in a second paper.

9. Discussion

Although the notion of fairness is inherently nonconstructive, such a straightforward approach via classical theories leads to smooth and somewhat surprising results. We have presented our first results here. Work is still in progress. More results involve possible hierarchies between edge- and path-fairness and letter- and word-fairness, a deeper connection between edge-fairness and ω-regularity and decision algorithms, and will be presented in a second paper.

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